NESA No. | | | | | | | | |

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2022

YEAR 12

AP4

MATHEMATICS EXTENSION 1

Time allowed – 2 hours plus 10 minutes reading time

General Instructions

- Attempt all questions
- Write your NESA number on the question paper
- Write using black pen
- NESA approved calculators may be used
- The NESA reference sheet has been provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 4)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 5 – 10)

- Attempt Questions 11-14
- Each question must be commenced in a new booklet clearly marked with your NESA number and question number eg. Question 11, Question 12, Question 13, or Question 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

1 A school committee consists of 8 members and a chairperson. The members are selected from 12 students. The chairperson is selected from 4 teachers. In how many ways could the committee be selected?

A. ${}^{12}C_8 + {}^{4}C_1$

B. ${}^{12}P_8 + {}^{4}P_1$

C. ${}^{12}P_8 \times {}^{4}P_1$

D. ${}^{12}C_8 \times {}^{4}C_1$

2 What is the value of k for which $\int_0^k \frac{1}{4+x^2}\,dx = \frac{\pi}{6}$?

A. 1

B. $\frac{1}{2}$

C. $\sqrt{3}$

D. $2\sqrt{3}$

3 The function $f(x) = \dfrac{x^2}{x^3 - 6}$ has an inverse $f^{-1}(x)$ for $x \geq 0$.

Which of the following represents a point of intersection of $f(x)$ with $f^{-1}(x)$?

A. $(2, 2)$

B. $(1, -0.2)$

C. $(3, 3)$

D. $(-0.2, 1)$

4 The position vector of a point A is given by $2\underset{\sim}{i} + 3\underset{\sim}{j}$.

The vector $\overrightarrow{OA}$ is rotated 90^0 clockwise to get $\overrightarrow{OB}$. What is $\overrightarrow{OB}$?

A. $3\underset{\sim}{i} - 2\underset{\sim}{j}$

B. $3\underset{\sim}{i} + 2\underset{\sim}{j}$

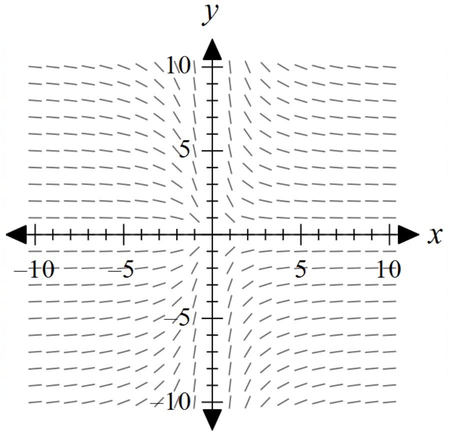
C. $-3\underset{\sim}{i} - 2\underset{\sim}{j}$

D. $-3\underset{\sim}{i} + 2\underset{\sim}{j}$

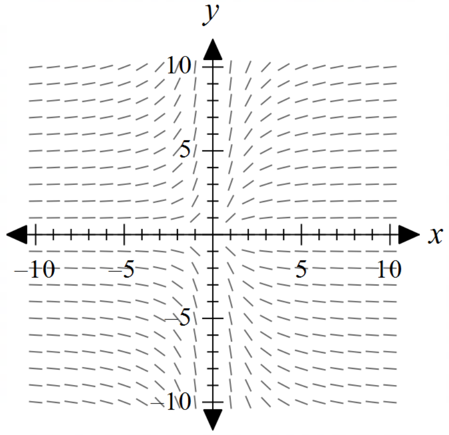
5 A differential equation is given to be $\dfrac{dy}{dx} = \dfrac{y}{x^2}$.

Which of the following best represents the direction field of the differential equation?

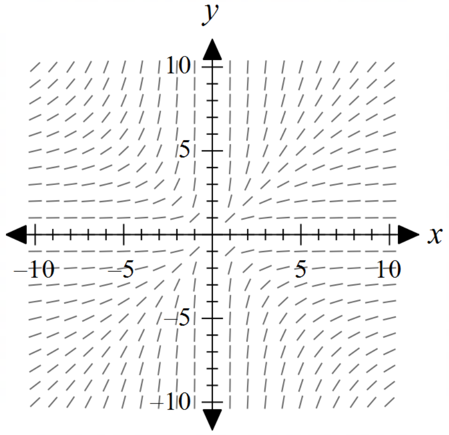
A.



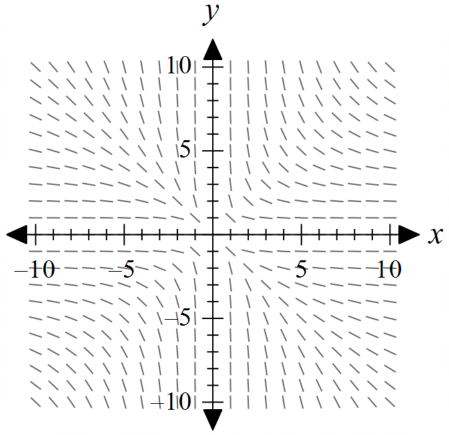
B.



C.



D.



6 How many solutions does the equation $\sin 6x - \sin 2x = 0$ have for $0 \leq x \leq 2\pi$?

A. 5

B. 12

C. 13

D. 14

7 What is the vector projection of $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j}$ onto $\underset{\sim}{v} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$?

A. $\frac{1}{5}(-3\underset{\sim}{i} + 4\underset{\sim}{j})$

B. $-\frac{1}{5}(-3\underset{\sim}{i} + 4\underset{\sim}{j})$

C. $\frac{1}{5}(\underset{\sim}{i} + 2\underset{\sim}{j})$

D. $\frac{1}{5}(\underset{\sim}{i} - 2\underset{\sim}{j})$

8 What is the range of the function with rule $f(x) = \cos^{-1}(2x - 1) + \frac{\pi}{2}$?

A. $[0, \pi]$

B. $[0, \frac{\pi}{2}]$

C. $[-\frac{\pi}{2}, \frac{3\pi}{2}]$

D. $[\frac{\pi}{2}, \frac{3\pi}{2}]$

9 What is the value of $\int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1} x \, dx}{\sqrt{1 - x^2}}$?

A. $\frac{\pi^2}{36}$

B. $\frac{\pi^2}{18}$

C. $\frac{\pi^2}{6}$

D. $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$

10 The parametric equations of a curve $\boldsymbol{C}$ are given by $x = \cos 2t$ and $y = \sin t$.
What is the cartesian equation of the curve $\boldsymbol{C}$?

A. $y^2 = \frac{1}{2}(x - 1)$

B. $y^2 = -\frac{1}{2}(x - 1)$

C. $y^2 = -\frac{1}{2}(x + 1)$

D. $y^2 = \frac{1}{2}(x + 1)$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Instructions

- Answer each question in the appropriate booklet. Extra writing booklets are available.
- In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Marks**

(a) Solve the inequality $\dfrac{3}{x-1} < 2$. **2**

(b) The letters of the word **EXAMINATION** are to be arranged in a row.

(i) How many eleven letter words can be made? **1**

(ii) How many eleven letter words will **NOT** contain adjacent N's? **2**

(c) The polynomial equation $x^4 + 4x^3 - 3x^2 + 5\text{x} - 7 = 0$ has roots α, β, γ and δ. **2**

Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$.

(d) Use the substitution $u = 2x - 1$ to evaluate $\displaystyle\int_{\frac{1}{2}}^{2} x\,\sqrt[3]{2x-1}\,dx$. **3**

Write your answer correct to one decimal place.

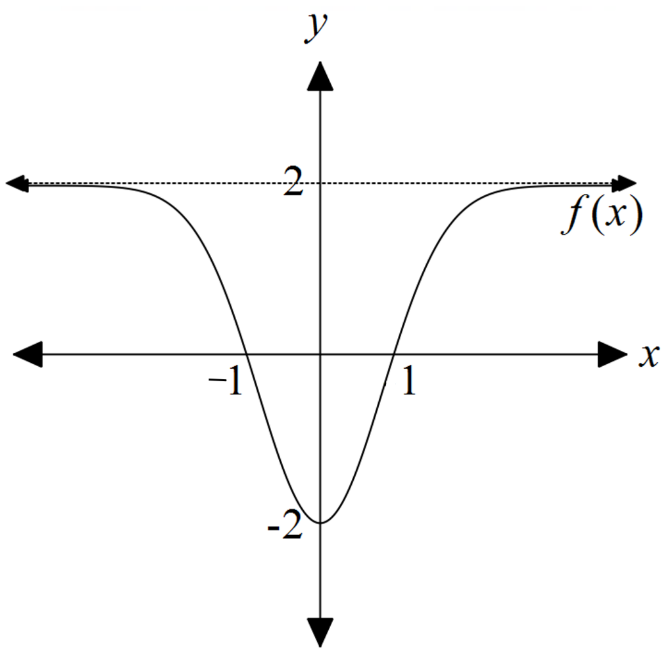
(e) (i) Sketch the curve $y = \sqrt{x^2 - 2x}$ for $x \in [2,4]$, clearly labelling the end points with their coordinates. **2**

(ii) The portion of the curve given by $y = \sqrt{x^2 - 2x}$ for $x \in [2,4]$ is rotated about the $x-$axis to form a solid of revolution. **3**
Evaluate the volume of the solid formed in exact form.

End of Question 11

Question 12 (15 marks) **Marks**

(a) The diagram shows the graph of $f(x)$, which has a y-intercept at -2, x-intercepts at 1 and -1, and a horizontal asymptote at $y = 2$. **3**



Sketch the graph of $y = \frac{1}{f(x)}$, showing any asymptotes and intercepts.

(b) Use the ***t*-formula** to solve the equation $\sin\theta + \cos\theta = \frac{1}{2}$ for $[0, 2\pi]$ **4**
Provide your answer in radians to three decimal places.

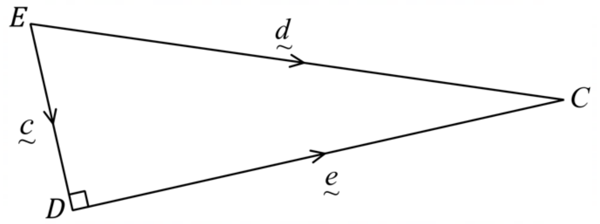
(c) Find the coefficient of x^8 in the expansion of $\left(\frac{x}{4} + \frac{4}{x}\right)^{14}$. **2**

(d) When the polynomial $P(x)$ is divided by $(x-1)(x+4)$, the quotient is $Q(x)$ and the remainder is $R(x)$. The remainder $R(x)$ can be written in the general form as $R(x) = ax + b$. **3**
It is known that $P(1) = -4$ and when $P(x)$ is divided by $(x+4)$, the remainder is 1.
Find the values of a and b.

Question 12 continues on page 7

Question 12 continued

(e) ΔDEC has a right angle at D.



Show that :

(i) $|\underset{\sim}{d}|^2 = \underset{\sim}{e} \cdot \underset{\sim}{e} + 2(\underset{\sim}{e} \cdot \underset{\sim}{c}) + \underset{\sim}{c} \cdot \underset{\sim}{c}$ **2**

(ii) $|\underset{\sim}{d}|^2 = |\underset{\sim}{e}|^2 + |\underset{\sim}{c}|^2$ **1**

End of Question 12

Question 13 (15 marks) **Marks**

(a) Eight students, including Ari, Bobbi, and Cali, form a single line and walk into their classroom.

(i) How many ways can the students walk into the classroom if Ari and Bobbi are next to each other, with no students in between? **1**

(ii) When the students enter the classroom, they are seated around a circular table. **2**

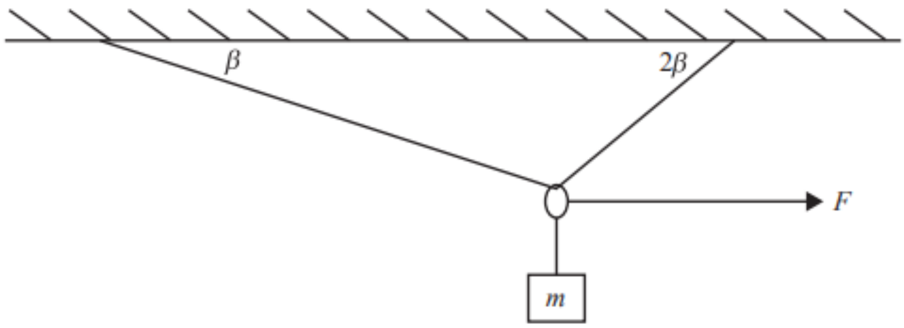
How many ways can the students be seated around the table if Bobbi and Cali cannot sit next to each other?

(b) An ice cube is melting at a rate of 2 mL/min. The three dimensions of the cube are decreasing uniformly. **2**

At what rate is the side length of the cube decreasing when the side length is 3 cm? Give your answer in cm/min. (1 mL = 1 cm^3)

(c) (i) Show that $\dfrac{\cos\beta - \cos 2\beta}{\sin\beta + \sin 2\beta} = \dfrac{1-\cos\beta}{\sin\beta}$ **2**

(ii) A light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs from a smooth ring on the string. A horizontal force of F newtons is applied to the string. The tension in the string has a constant magnitude and the system is in equilibrium. At one end string makes an angle β with the ceiling and at the other end the string makes an angle 2β, as shown in the diagram below. **3**



Show that $F = mg\left(\dfrac{1-\cos\beta}{\sin\beta}\right)$.

Question 13 continues on page 9

Question 13 continued

(d) The vector equation of the velocity $\underset{\sim}{v}(t)$ of a particle at any time t is given by:

$$\underset{\sim}{v}(t) = 10\sqrt{3}\,\underset{\sim}{i} - (10 + 10\,t\,)\underset{\sim}{j}$$

Initially the particle was projected from the edge of a cliff 120 metres high. Assuming that the acceleration due to gravity is 10 m/s^2 and the foot of the cliff as the origin:

(i) Show that the position vector of the particle at any time $\boldsymbol{t}$ is given by $\underset{\sim}{r}(t) = 10\sqrt{3}\,t\,\underset{\sim}{i} + (120 - 10t - 5t^2)\underset{\sim}{j}$ metres **2**

(ii) Calculate the time taken by the particle to reach the ground. **1**

(iii) Hence, find the speed and the angle at which the particle hits the ground. **2**

End of Question 13

Question 14 (15 marks) **Marks**

(a) A new virus is spreading through a closed population. The rate of change in the number of infected individuals can be given by the logistic equation
$\frac{dN}{dt} = kN(P - N)$,
where N is the number of infected individuals, t is the time in days, P is the constant population size, and k is a constant.

(i) Show that $\frac{1}{N(P-N)} = \frac{1}{P}\left(\frac{1}{N} + \frac{1}{P-N}\right)$. **1**

(ii) By integrating the differential equation, show that, $N = \frac{P}{1 + Be^{-Pkt}}$, where B is a constant **3**

(iii) If there is initially one infected individual in the population of 1,000,000 individuals, and after 28 days there are 100 infected individuals, find how many individuals are infected 12 weeks after the initial infection. **3**

(b) Find the two possible values of k for which the polynomial $P(x) = x^3 + 3x^2 - 45x + k$ has a double zero. **3**

(c) (i) Show algebraically that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. **2**

(ii) Hence, prove by mathematical induction that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$ **3**

for $n \geq 1$.

End of Paper

Year 12 Mathematics Extension 1 AP4 Section I – Answer Sheet

NESA No. | | | | | | | | | |

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

A ● B ⊗ correct C ○ D ○

---

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

1 $^{12}C_8 \times {}^4C_1$

D

2 $\int_0^K \frac{dx}{4+x^2} = \frac{\pi}{6}$ D

$\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^K = \frac{\pi}{6}$

$\frac{1}{2}\tan^{-1}\frac{K}{2} - \frac{1}{2}\tan^{-1}0 = \frac{\pi}{6}$

$\frac{1}{2}\tan^{-1}\frac{K}{2} = \frac{\pi}{6}$

$\tan^{-1}\frac{K}{2} = \frac{\pi}{3}$

$\frac{K}{2} = \tan\frac{\pi}{3}$

$\frac{K}{2} = \sqrt{3}$

$K = 2\sqrt{3}$

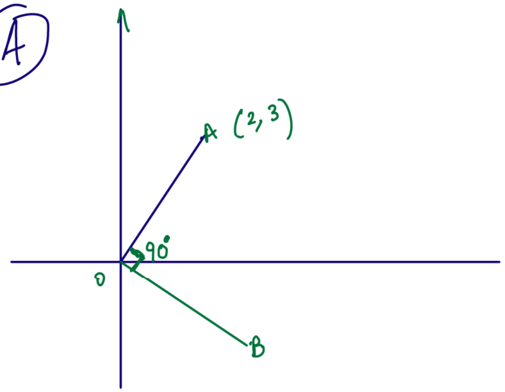
3. A function may only intersect its inverse on $y = x$.

$$f(2) = \frac{2^2}{2^3 - 6} = 2.$$

$(2, 2)$ lies on the function.

$\therefore$ A

4 A



$\vec{OB} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$

5. B

6. C $\sin 6x - \sin 2x = 0$

$2 \cos 4x \sin 2x = 0$

$\cos 4x = 0$ twice for each cycle around the unit circle, so $2 \times 4 = 8$ times for $0 \le 4x \le 8\pi$.

$\sin 2x = 0$ three times for each cycle around the unit circle, so $3 \times 2 = 6$ times

However, Since 2π gets repeated twice for $0 \le 2x \le 4\pi$ $\therefore$ no. of solutions $= 5$

So, there are $8 + 5 = 13$ solutions.

7. A $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j}$ $\quad$ $\underset{\sim}{v} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$

$$\text{proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v} = \frac{1 \cancel{5}}{\cancel{25}\, 5}(-3\underset{\sim}{i} + 4\underset{\sim}{j}) = \frac{1}{5}(-3\underset{\sim}{i} + 4\underset{\sim}{j})$$

where $\underset{\sim}{u} \cdot \underset{\sim}{v} = (\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot (-3\underset{\sim}{i} + 4\underset{\sim}{j})$

$= -3 + 8$

$= 5$

and $|\underset{\sim}{v}|^2 = \left(\sqrt{(-3)^2 + (4)^2}\right)^2 = 25$

8. D

Since Range of $\cos^{-1}(2x-1)$ is between 0 & π

$\therefore$ Range of $\cos^{-1}(2x-1)+\frac{\pi}{2}$ is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

9. B

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$$

as $\int f'(x) \times f(x)\,dx$

$= \frac{f(x)^{n+1}}{2} + C$

let $f(x) = \sin^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

$$\left[\frac{1}{2}(\sin^{-1}x)^2\right]_0^{\frac{\sqrt{3}}{2}}$$

$$\frac{1}{2}\left(\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)^2 - \left(\sin^{-1}0\right)^2\right)$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{3}\right)^2 - 0^2\right) = \frac{\pi^2}{2\times 9} = \frac{\pi^2}{18}$$

10. B.

$x = \cos 2t$

$x = 1 - 2\sin^2 t$

$y = \sin t$

$\therefore x = 1 - 2y^2$

$2y^2 = 1 - x$

$y^2 = \frac{1-x}{2}$

$y^2 = -\frac{1}{2}(x-1)$

11a) $\frac{3}{x-1} < 2$

$\frac{3}{(x-1)}(x-1)^2 < 2(x-1)^2$

$3(x-1) < 2(x-1)^2$

$3(x-1) - 2(x-1)^2 < 0$

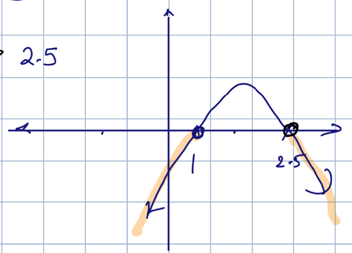
$(x-1)\left(3-2(x-1)\right) < 0$

$(x-1)(3-2x+2) < 0$

$(x-1)(5-2x) < 0$

$x < 1$ or $x > 2.5$

(1 mark)



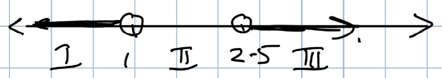
Alternate Method.

Critical Point $x = 1$

let $\frac{3}{x-1} = 2$

$\therefore x = 2.5$.

for $x=0$ $\frac{3}{0-1} = -3 < 2$ ✓



for $x = 2$ $\frac{3}{2-1} = 3 < 2$ ✗

$\therefore x < 1$ or $x > 2.5$

$x = 3$ $\frac{3}{3-1} = 1.5 < 2$ ✓

11 b) (i) E X A M I N A T I O N

1 2 3 4 5 6 7 8 9 10 11

$\frac{11!}{2!\,2!\,2!} = 4989600$

(ii). $\boxed{\cdot}E\boxed{\cdot}X\boxed{\cdot}A\boxed{\cdot}M\boxed{\cdot}I\boxed{\cdot}A\boxed{\cdot}T\boxed{\cdot}I\boxed{\cdot}O\boxed{\cdot}$

N can be inserted into these blank boxes in $\frac{^{10}P_2}{2!}$ ways.

and the rest of the letters can be arranged in $\frac{9!}{2!\times 2!}$ ways

$\therefore$ Total No. of ways $= \frac{^{10}P_2}{2!} \times \frac{9!}{2!\times 2!} = 4082400$

Alternate Method.

If we combine two N's then all letters can be arranged in $\frac{10!}{2!\times 2!}$

$\therefore$ No. of ways when N's are not adjacent = Total no. of ways $- \frac{10!}{2!\times 2!}$

$= \frac{11!}{2!\,2!\,2!} - \frac{10!}{2!\times 2!} = 4082400$

c) $x^4 + 4x^3 - 3x^2 + 5x - 7 = 0$

$\alpha + \beta + \gamma + \delta = -4$

$\alpha\beta\gamma + \beta\gamma\delta + \alpha\beta\delta + \alpha\gamma\delta = -5$

$\alpha\beta\gamma\delta = -7$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

$$\frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{-5}{-7} = \frac{5}{7}$$

11d) $$\int_{1/2}^{2} x\sqrt[3]{2x-1}\,dx$$

let $u = 2x - 1 \Rightarrow \frac{du}{dx} = 2$

when $x = \frac{1}{2}$, $u = 0$; when $x = 2$, $u = 3$

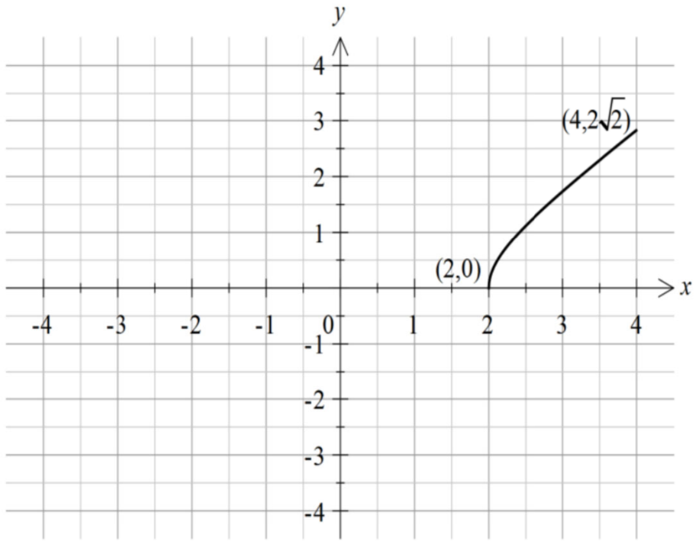
$x = \frac{u+1}{2}$

$$\int_0^3 \frac{(u+1)(u)^{\frac{1}{3}}}{2\times 2}\,du = \int_0^3 \frac{\left(u^{\frac{1}{3}}\times u + u^{\frac{1}{3}}\right)}{4}\,du$$

$$\int_0^3 \frac{u^{\frac{4}{3}} + u^{\frac{1}{3}}}{4}\,du = \frac{1}{4}\left[\frac{3u^{\frac{7}{3}}}{7} + \frac{3u^{\frac{4}{3}}}{4}\right]_0^3$$

$$= \frac{1}{4}\left(\frac{3}{7}(3)^{\frac{7}{3}} + \frac{3}{4}(3)^{4/3}\right) \approx 2.2.$$

11e(i)



11e(ii) Volume of the solid formed $= \int_2^4 \pi y^2 \, dx$

$$= \int_2^4 \pi(x^2 - 2x)\, dx$$

$$= \left[\pi \frac{x^3}{3} - \pi \times \frac{\cancel{2}x^2}{\cancel{2}}\right]_2^4$$

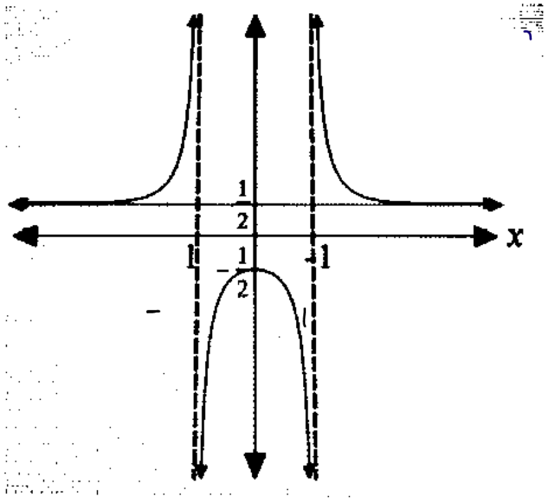
$$= \left(\frac{(4)^3}{3}\pi - 4^2\pi\right) - \left(\frac{\pi(2)^3}{3} - \pi(2)^2\right)$$

$$= \left(\frac{64\pi}{3} - 16\pi\right) - \left(\frac{8\pi}{3} - 4\pi\right)$$

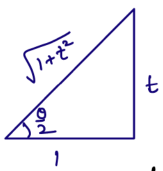
$$= \frac{64\pi}{3} - 16\pi - \left(-\frac{4\pi}{3}\right)$$

$$= \frac{64\pi - 48\pi + 4\pi}{3} = \frac{20\pi}{3}$$ cubic units.

12 a



12 let $t = \tan\frac{\theta}{2}$



$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2 \times \frac{t}{\sqrt{1+t^2}} \times \frac{1}{\sqrt{1+t^2}}$$

$$= \frac{2t}{1+t^2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2}$$

$$\sin\theta + \cos\theta = \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}$$

$$\frac{1+2t-t^2}{1+t^2} = \frac{1}{2}$$

$$2(1+2t-t^2) = 1+t^2$$

$$3t^2-4t-1 = 0$$

$$t = \frac{4 \pm \sqrt{16-4(3)(-1)}}{6}$$

$$t = \frac{4 \pm \sqrt{28}}{6}$$

$$t = \frac{2 \pm \sqrt{7}}{3}$$

$$\tan\frac{\theta}{2} = \frac{2 \pm \sqrt{7}}{3}, \quad 0 < \theta \le 2\pi \rightarrow 0 < \frac{\theta}{2} \le \pi$$

$$\frac{\theta}{2} = \tan^{-1}\frac{2 \pm \sqrt{7}}{3}$$

$$\frac{\theta}{2} = 0.997, \ \pi - 0.212$$

$$\frac{\theta}{2} = 0.997, \ 2.930 \text{ rad}$$

$$\theta = 1.995, \ 5.859 \text{ rad}$$

12(c)

$$T_{k+1} = {}^{14}C_k \left(\frac{x}{4}\right)^{14-k} \left(\frac{4}{x}\right)^{k}$$

$$T_{k+1} = {}^{14}C_k \frac{(4)^k}{(4)^{14-k}} (x)^{14-2k}$$

$$14-2k = 8$$
$$2k = 14-8$$
$$2k = 6$$
$$k = 3$$

$$T_4 = {}^{14}C_3 \frac{(4)^3}{(4)^{14-3}}$$

$$= \frac{{}^{14}C_3}{(4)^8} = \frac{91}{16384}$$

12(d).

$$P(x) = (x-1)(x+4)\,Q(x) + R(x)$$

$$P(x) = (x-1)(x+4)Q(x) + ax + b \longrightarrow 1$$

$$P(1) = -4$$

$$P(1) = 0 + a\times 1 + b$$
$$a + b = -4 \longrightarrow 2$$

Substituting $x=-4$ in 1

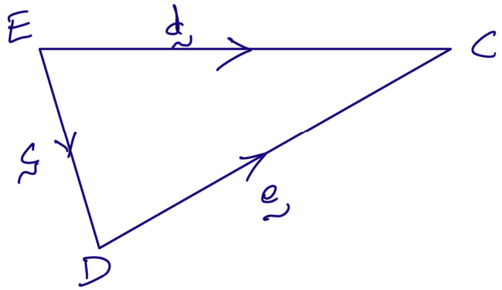
$$P(-4) = a\times -4 + b$$
$$-4a + b = 1 \longrightarrow 3$$

$$a + \cancel{b} = -4$$
$$\underset{(+)}{-}4a \,\underline{+}\, b = \underline{\ }1$$
$$5a = -5$$
$$a = -1$$

$$a + b = -4$$
$$-1 + b = -4$$
$$b = -4+1$$
$$= -3$$

$\therefore \; R(x) = -1x - 3$
$= -x - 3$

12(e)



(i) $\underset{\sim}{d} = \underset{\sim}{c} + \underset{\sim}{e}$

$|\underset{\sim}{d}|^2 = \underset{\sim}{d} \cdot \underset{\sim}{d}$

$= (\underset{\sim}{e} + \underset{\sim}{c}) \cdot (\underset{\sim}{e} + \underset{\sim}{c})$

$= \underset{\sim}{e} \cdot \underset{\sim}{e} + \underset{\sim}{e} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{e} + \underset{\sim}{c} \cdot \underset{\sim}{c}$

Since $\underset{\sim}{c} \cdot \underset{\sim}{e} = \underset{\sim}{e} \cdot \underset{\sim}{c}$

$= \underset{\sim}{e} \cdot \underset{\sim}{e} + 2\underset{\sim}{e} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c}$

(ii). Vectors $\underset{\sim}{e}$ and $\underset{\sim}{c}$ are perpendicular

$\therefore \; \underset{\sim}{e} \cdot \underset{\sim}{c} = 0$

$|\underset{\sim}{d}|^2 = \underset{\sim}{e} \cdot \underset{\sim}{e} + 2 \times 0 + \underset{\sim}{c} \cdot \underset{\sim}{c}$

$= |\underset{\sim}{e}|^2 + |\underset{\sim}{c}|^2$

13a.

(i). If we join Ari & Bobbi as a single unit, then they can be arranged amongst themselves in 2! ways.

Since, there are seven different units which can be arranged in 7! ways.

∴ the total number of ways $= 7! \times 2!$
$= 10080$

(ii) Ways for 8 students to be seated around a circular table $(8-1)! = 5040$

Ways for Bobbi and Cali to sit next to each other :- $(7-1)! \times 2! = 1440$

Ways for the students to be seated with Bobbi & Cali separated .$= 5040 - 1440$
$= 3600$

13b

$$V = s^3$$

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$$

$$\therefore \frac{ds}{dt} = \frac{1}{3} s^{-2} \frac{dV}{dt}$$

When $s = 3$

$$\frac{ds}{dt} = \frac{1}{3}(3)^{-2}(-2) = \frac{-2}{27} \approx -0.074 \text{ cm/min}$$

13 c

(i) L.H.S

$$\frac{\cos\beta - \cos 2\beta}{\sin\beta + \sin 2\beta} = \frac{\cos\beta - (2\cos^2\beta - 1)}{\sin\beta + 2\sin\beta\cos\beta}$$

$$= \frac{\cos\beta - 2\cos^2\beta + 1}{\sin\beta(1 + 2\cos\beta)}$$

$$= \frac{-2\cos^2\beta + \cos\beta + 1}{\sin\beta(1 + 2\cos\beta)}$$

$$= \frac{-2\cos^2\beta + 2\cos\beta - \cos\beta + 1}{\sin\beta(1 + 2\cos\beta)}$$

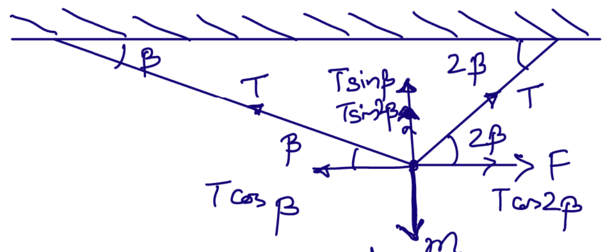
$$= \frac{-2\cos\beta(\cos\beta - 1) - 1(\cos\beta - 1)}{\sin\beta(1 + 2\cos\beta)}$$

$$= \frac{(2\cos\beta - 1)(\cos\beta - 1)}{\sin\beta(1 + 2\cos\beta)}$$

$$= -\frac{\cancel{(2\cos\beta + 1)}(\cos\beta - 1)}{\sin\beta\cancel{(1 + 2\cos\beta)}}$$

$$= \frac{-(\cos\beta - 1)}{\sin\beta} = \frac{1 - \cos\beta}{\sin\beta}$$

13 c (ii).



Equating the horizontal components

$$T\cos 2\beta + F = T\cos\beta$$

$$\therefore F = T\cos\beta - T\cos 2\beta$$

$$F = T(\cos\beta - \cos 2\beta) \longrightarrow (1)$$

Equating the vertical components:

$$T\sin\beta + T\sin 2\beta = mg$$

$$\text{or} \quad T = \frac{mg}{\sin\beta + \sin 2\beta} \rightarrow (2)$$

Substituting the value of T from (2) into (1)

$$F = \frac{mg}{(\sin\beta + \sin 2\beta)}(\cos\beta - \cos 2\beta)$$

$$= mg\left(\frac{\cos\beta - \cos 2\beta}{\sin\beta + \sin 2\beta}\right)$$

Using the result from part (i),

$$F = mg\left(\frac{1-\cos\beta}{\sin\beta}\right)$$

13 d (ii)

$$\underset{\sim}{v}(t) = 10\sqrt{3}\,\underset{\sim}{i} - (10+10t)\underset{\sim}{j}$$

$$\dot{x}(t) = 10\sqrt{3}$$

$$\int_{x(0)}^{x(t)} \dot{x}(t)\,dt = \int_0^t 10\sqrt{3}\,dt$$

$$x(t) - x(0) = 10\sqrt{3}t$$

$$x(0) = 0$$

$$x(t) = 10\sqrt{3}t$$

$$\dot{y}(t) = -(10+10t)$$

$$\int_{y(0)}^{y(t)} \dot{y}(t)\,dt = \int_0^t -(10+10t)\,dt$$

$$y(t) - y(0) = -10t - \frac{\cancel{10}^{5}t^2}{\cancel{2}}$$

$$y(0) = 120$$

$$y(t) = -10t - 5t^2 + 120$$

$$\therefore \underset{\sim}{r}(t) = 10\sqrt{3}t\,\underset{\sim}{i} + (120 - 10t - 5t^2)\underset{\sim}{j}$$

13d (ii) $y(t) = 0$ when the particle reaches the ground

$$120 - 10t - 5t^2 = 0$$

$$-5t^2 - 10t + 120 = 0$$

$$-5(t^2 + 2t - 24) = 0$$

$$t^2 + 6t - 4t - 24 = 0$$

$$(t+6)(t-4) = 0$$

$t = -6$ or $t = 4$

Discard $t = -6$

$\therefore$ the particle reaches the ground when $t = 4$

13d(iii) $V_x(t) = 10\sqrt{3}$ m/s

Horizontal component velocity when the particle reaches the ground

Vertical Component

$$V_y(4) = -(10 + 10 \times 4)$$

$$= -(10 + 40)$$

$$= -50 \text{ m/s}$$

$$\text{Speed} = \sqrt{V_x^2 + V_y^2}$$

$$= \sqrt{(10\sqrt{3})^2 + (-50)^2}$$

$$= 52.92 \text{ m/s}$$

$$\tan\theta = \frac{-50}{10\sqrt{3}}$$

$$\theta \approx -70^\circ 54'$$

14a (i). $R.H.S = \frac{1}{P}\left(\frac{1}{N} + \frac{1}{P-N}\right)$

$$= \frac{1}{P}\left(\frac{P - \cancel{N} + \cancel{N}}{N(P-N)}\right)$$

$$= \frac{1}{\cancel{P}} \times \frac{\cancel{P}}{N(P-N)} = \frac{1}{N(P-N)} = L.H.S$$

Hence Proved.

14a(ii).

$$\frac{dN}{dt} = kN(P-N)$$

$$\frac{1}{N(P-N)}\,dN = k\,dt$$

$$\frac{1}{P}\left(\frac{1}{N}+\frac{1}{P-N}\right)dN = k\,dt$$

$$\int\left(\frac{1}{N}+\frac{1}{P-N}\right)dN = \int Pk\,dt$$

$$\ln(N) - \ln(P-N) = Pkt + c$$

$$\left(\frac{1}{N} > 0 \text{ and } \frac{1}{P-N} > 0\right)$$

$$\ln\frac{N}{P-N} = Pkt + c$$

$$\frac{N}{P-N} = e^{Pkt+c}$$

$$= e^{c}e^{Pkt} \qquad (\text{let } A = e^{c})$$

$$\frac{N}{P-N} = Ae^{Pkt}$$

$$N = (P-N)Ae^{Pkt}$$

$$N = PAe^{Pkt} - NAe^{Pkt}$$

$$N + NAe^{Pkt} = PAe^{Pkt}$$

$$N(1+Ae^{Pkt}) = PAe^{Pkt}$$

$$N = \frac{PAe^{Pkt}}{1+Ae^{Pkt}}$$

Multiply numerator and denominator by e^{-Pkt}

$$N = \frac{PA}{e^{-Pkt}+A}$$

$$N = \frac{P}{\frac{1}{A}e^{-Pkt}+1}$$

$$= \frac{P}{1+Be^{-Pkt}} \qquad \text{where } B = \frac{1}{A}$$

(iii) $P = 1000000$, when $t=0$, $N=1$

$$1 = \frac{1000000}{1+B}$$

$$B = 999999$$

When $t = 28$, $N = 100$

$$100 = \frac{1000000}{1 + 999999 e^{-(1000000)K(28)}}$$

$$1 + 999999 e^{-28000000K} = 10000$$

$$e^{-28PK} = \frac{101}{10101}$$

$$PK \approx 0.164474$$

Find N when $t = 84$

$$N = \frac{1000000}{1 + 999999 e^{-0.164474(84)}}$$

$$= 500077 \text{ (nearest individual)}$$

14 b). let the double zero exists for $x = \alpha$.

$P(\alpha) = 0$

$\alpha^3 + 3\alpha^2 - 45\alpha + k = 0$

$P'(x) = 3x^2 + 6x - 45$

$P'(\alpha) = 3\alpha^2 + 6\alpha - 45$

$3\alpha^2 + 6\alpha - 45 = 0$

$3(\alpha^2 + 2\alpha - 15) = 0$

$\alpha^2 + 2\alpha - 15 = 0$

$(\alpha + 5)(\alpha - 3) = 0$

$\alpha = -5$ or $\alpha = 3$

if $\alpha = -5$ is a double zero then

$P(-5) = 0$

$(-5)^3 + 3(-5)^2 - 45(-5) + k = 0$

$-125 + 25 \times 3 + 225 + k = 0$

$k + 175 = 0$

$\therefore k = -175$

If $\alpha = 3$ is a double zero then $P(3) = 0$

$(3)^3 + 3(3)^2 - 45 \times 3 + k = 0$

$27 + 27 - 135 + k = 0$

$k = 81$.

Can also be done alternatively by using α, α, β as the roots of the equation.

14 c (i) RHS.

$$\binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\frac{(n-1)!}{(r-1)!\,(n-\cancel{1}-r+\cancel{1})!} + \frac{(n-1)!}{(n-1-r)!\,r!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{(n-r-1)!\,r!}$$

$$= \frac{(n-1)!}{(r-1)!(n-r)(n-r-1)!} + \frac{(n-1)!}{(n-r-1)!\,r\,(r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left[\frac{1}{n-r} + \frac{1}{r}\right]$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left(\frac{\cancel{r} + n - \cancel{r}}{(n-r)(r)}\right)$$

$$= \frac{(n-1)! \times n}{(r-1)!\,(n-r-1)!\,(n-r)\,r}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^{n}C_{r} = L.H.S.$$

14c

(ii). Let $n = 1$

$$L.H.S = \binom{1}{0} + \binom{1}{1} = 2$$

$$R.H.S = 2^{1} = 2.$$

L.H.S = R.H.S, so it is true for $n=1$

Assume it to be true for $n=k$

$$\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k}=2^k$$

let $n=k+1$

So, need to show

$$\binom{k+1}{0}+\binom{k+1}{1}+\binom{k+1}{2}+\dots+\binom{k+1}{k}+\binom{k+1}{k+1}=2^{k+1}$$

L.H.S

$$\binom{k+1}{0}+\binom{k+1}{1}+\binom{k+1}{2}+\dots+\binom{k+1}{k}+\binom{k+1}{k+1}$$

$$\binom{k+1}{0}+\binom{k}{0}+\binom{k}{1}+\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k-1}+\binom{k}{k}+\binom{k+1}{k+1}$$

Since $\binom{k+1}{0}=\binom{k}{0}$

and $\binom{k+1}{k+1}=\binom{k}{k}$

$$\therefore \binom{k}{0}+\binom{k}{0}+\binom{k}{1}+\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k-1}+\binom{k}{k}+\binom{k}{k}$$

$$2\left(\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+\dots+\binom{k}{k}\right)$$

$$2\times 2^k=2^{k+1}$$

$\therefore$ By the principle of mathematical induction, the statement is true.